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Gaussian, Non-Gaussian Critical Fluctuations in the Curie–Weiss Model

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It is known that at the critical temperature the Curie–Weiss mean-field model has non-Gaussian fluctuations and that "internal fluctuations" can be Gaussian. Here we compute the distribution of the q-mode magnetization fluctuations as a function of the temperature, the wave vector q, and a fading out external field. We obtain new classes of probability distributions generated by this external field as well as new critical behavior in terms of its rate of fading out. We discuss also the susceptibility as the limit q tending to zero.

KEY WORDS: Curie–Weiss model; critical probability distributions; susceptibility; critical external fields; modulated fluctuations.

1. INTRODUCTION

In the classical Curie–Weiss mean-field model of ferromagnetism and in fact for all mean-field type of interactions, it was shown some time ago[1,2] that one has non-Gaussian critical fluctuation distributions. The situation for quantum mean-field models is not different.[3] This is a manifestation of the critical behavior expressed in terms of probabilistic results.

More recently[4] the presence of a substantial Gaussian element at the critical point was put forward, by showing that "fluctuations within the fluctuating field are Gaussian," as the statement was formulated. Our attention was attracted by this result and we were interested in understanding whether this was an isolated property or whether this result could be a

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special case of something more general. The results of this paper indicate that indeed there is a rich structure behind it.

In order to make the paper more accessible, we work out our results explicitly for the classical Curie–Weiss model, although up to details everything can be performed for more general systems.[5] As is well known, the spins of the model are represented by the random variables $\sigma \equiv \{\sigma_x = \pm 1\}_{x \in \Lambda}$, $\Lambda = \{1, 2,..., N\}$ in the Gibbs distribution $\mu^{\Lambda}_{\beta,h}(\cdot)$, defined by the Hamiltonian

$$H_\Lambda(\sigma, h) = -\frac{J}{2N} \sum_{x,y \in \Lambda} \sigma_x \sigma_y - \sum_{x \in \Lambda} h_x \sigma_x$$

where $h_x \in \mathbb{R}$, $J > 0$; i.e., the finite volume Λ measure is given by, for all $\beta = 1/kT \in \mathbb{R}^+$,

$$\mu^{\Lambda}_{\beta,h}(X) = \frac{1}{Z_\Lambda(\beta, h)} \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda}} e^{-\beta H_\Lambda(\sigma, h)} X$$

where $X: \sigma \to \mathbb{R}$ and

$$Z_\Lambda(\beta, h) = \sum_{\substack{\sigma_x = \pm 1 \\ x \in \Lambda}} e^{-\beta H_\Lambda(\sigma, h)}$$

is the partition function; h_x is a pointwise magnetic field

The partition function is easily computed to be

$$Z_\Lambda(\beta, h) = \left(\frac{N}{2\pi}\right)^{1/2} \int dv \exp\left(-\frac{Nv^2}{2} + \sum_{x \in \Lambda} \ln\{2ch[v(\beta J)^{1/2} + \beta h_x]\}\right) \tag{1}$$

We are interested in the limit distribution of the random variable

$$F^q_{\delta,\Lambda}(\sigma) = \frac{1}{N^{1/2+\delta}} \sum_{x \in \Lambda} (\sigma_x - m_\Lambda) \cos qx; \qquad q \in [0, 2\pi) \tag{2}$$

i.e., in the q-mode fluctuation of the magnetization; m_Λ is the magnetization in the Gibbs measure

$$m_\Lambda = \mu^{\Lambda}_{\beta,h}(\sigma_x)$$

In this paper we compute the characteristic function of the distribution of the random variable $F^q_{\delta,\Lambda}(\sigma)$:

$$\phi(t; \beta, q, h) = \lim_{N \to \infty} \mu^{\Lambda}_{\beta,h}(\exp[itF^q_{\delta,\Lambda}(\sigma)]) \tag{3}$$

We are looking for a value δ such that this random variable has a non-trivial limiting distribution.

This limit is studied in terms of the three parameters β or the temperature T, the wave vector q, and the external field h.

As far as the T dependence is concerned, there is not so much new in the sense that there exists a critical temperature $T_c = J/k > 0$. We reproduce the well-known normal Gaussian limit distribution for $T > T_c$ for all q and h. We are essentially interested in new probability distributions in the critical situation, i.e., for $T = T_c$.

In Section 2 we consider first the limit distribution (3) as a function of q, but with vanishing magnetic field $h_x = 0$. We get a general result which yields as particular cases the results of refs. 1, 2, and 4 for very special values of q, i.e., we obtained previous results in a unified scheme. Our results give a clear explanation for the appearance of Gaussian "internal fluctuations."

In Section 3 we go one step further and we consider also the influence of the magnetic field as a boundary condition, i.e., we put $h_x = \hat{h}/N^\alpha$, $\alpha > 0$. This means that h_x tends to zero when the number of sites increases to infinity. At criticality it produces many new distributions for $F_\delta^q(\sigma)$ and different critical indices δ: $\delta = \alpha/3$ for $\alpha < 3/2$, $\delta = 1/4$ for $\alpha > 3/2$.

In Section 4 we discuss the long-wavelength limit $q = \hat{q}/N^\gamma$, $\gamma > 0$. The joint limits $h_x \to 0$ and $q \to 0$ lead again to new distributions and a new dependence of the index δ on the parameters α and γ.

We call a limit distribution normal if $\delta = 0$ and abnormal if $\delta > 0$; we call a distribution Gaussian if the characteristic function is of Gaussian type, i.e., quadratic in the exponential, even if $\delta \neq 0$. Our main results are that for the q-mode fluctuations of the magnetization (2) we obtain as a function of the parameters q and h the following distributions: normal and Gaussian (Theorems 2.1, 2.2, 3.1, and 3.2), abnormal and Gaussian (Theorem 3.3), and abnormal non-Gaussian (Theorems 2.2, 3.4, and 3.5). Concerning these different types of probability distributions, apart from those of refs. 1, 2, and 4, some of them did show up in the physical literature, e.g., in refs. 2 and 6 in the context of critical fluctuations and probability theory, and they were looked upon as "spurious." In the present analysis, a more unified presentation explains their occurrence in terms of the wave vector q and the magnetic field. We mention also ref. 7, where for spin glasses a so-called chaotic size dependence is discussed, indicating different types of probability laws.

We find also a particular value $\alpha_c = 3/2$ for the parameter α, describing the vanishing of the field h_x if N tends to infinity. For $\alpha > \alpha_c$, the influence of the field is nonexistent. For $\alpha \leqslant \alpha_c$, the presence of h determines the degree of criticality δ as well as the distribution, i.e., at α_c one gets a new type of transition for the critical fluctuations driven by α.

In the long-wavelength limit (Section 4) we discovered a regime for which the magnetization fluctuation only exists for subsequences of the increasing number N, indicating an essential singularity for the long-wavelength susceptibility.

2. ZERO MAGNETIC FIELD ($h=0$)

If $h=0$ in formula (1), then the partition function can be written as

$$Z_\Lambda(\beta, 0)=\left(\frac{N}{2\pi}\right)^{1/2} 2^N \int dv \exp\{-Nf(v, \beta)\}$$

where f is the free energy density or the rate function in the Laplace method:

$$\begin{aligned} f(v, \beta) &= \frac{v^2}{2} - \ln ch\, v(\beta J)^{1/2} \\ &= \frac{1}{2}(1-\beta J)\, v^2 + \frac{1}{12}(\beta J)^2 v^4 + O(v^6) \end{aligned}$$

Denote by β_c the value $\beta_c = 1/J$; it corresponds to the inverse critical temperature $T_c = 1/k\beta_c$. For $T > T_c$

$$f(v, \beta) = \tfrac{1}{2}(1-\beta J)\, v^2 + O(v^4) \tag{4}$$

and for $T = T_c$

$$f(v, \beta_c) = \tfrac{1}{12} v^4 + O(v^6) \tag{5}$$

Let φ be any continuous function on $\mathbb{R}$; we compute

$$\lim_N \int d\mu^\Lambda_{\beta,0}(v)\, \varphi(v), \qquad \text{with} \quad d\mu^\Lambda_{\beta,0}(v) = \frac{e^{-Nf(v,\beta)}\, dv}{\int e^{-Nf(v,\beta)}\, dv}$$

Then, using (4) or (5) and the standard Laplace argument, one gets

$$\lim_N \int d\mu^\Lambda_{\beta,0}(v)\, \varphi(v) = \varphi(0)$$

Denote for $q \in [0, 2\pi)$

$$a_{n,N}(q) = \frac{1}{N} \sum_{x \in \Lambda} \cos^n qx \tag{6}$$

and denote

$$a_n(q) = \lim_{N \to \infty} a_{n,N}(q); \qquad n = 1, 2, 3,... \tag{7}$$

It is important to remark that one can distinguish essentially two choices of the "wave vector" q: $a_1(q) = 0$ (i.e., $q \neq 0$) and $a_1(q) \neq 0$ (i.e., $q = 0$). In spite of the fact that the values of $a_{2k+1}(q) = \delta_{q,0}$ are evident, we prefer to keep the notation $a_n(q)$ in the formulas, in order to make clear the derivation and the results, as well as to suggest how the results can be extended to the case that the wave vector q depends on the volume N, a situation in which physicists are often interested (see Section 4).

Now we compute the characteristic function (3) for $h = 0$, i.e., for $m_\Lambda = 0$:

$$\begin{aligned}\phi(t, \beta, q, 0) &= \lim_N \mu^\Lambda_{\beta,0}(\exp[itF^q_{\delta,\Lambda}(\sigma)]) \\ &= \lim_N Z_\Lambda^{-1}(\beta, 0)\left(\frac{N}{2\pi}\right)^{1/2} \int dv \\ &\quad \times \exp\left\{-\frac{Nv^2}{2} + \sum_{x \in \Lambda} \ln 2\, ch[v(\beta J)^{1/2} + itN^{-(1/2+\delta)} \cos qx]\right\}\end{aligned}$$

Using (6), one gets

$$\begin{aligned}\phi(t, \beta, q, 0) = \lim_N \int d\mu^\Lambda_{\beta,0}(v) \exp\Big\{ & th[v(\beta J)^{1/2}]\, itN^{1/2-\delta} a_{1,N}(q) \\ & + ch^{-2}[v(\beta J)^{1/2}] \frac{1}{2}\left(\frac{it}{N^\delta}\right)^2 a_{2,N}(q) \\ & + \frac{(it)^3}{3!} N^{-3\delta} N^{-1/2} a_{3,N}(q)[ch^{-2} v(\beta J)^{1/2}]' + \cdots \Big\}\end{aligned} \tag{8}$$

Suppose now that $q \neq 0$, i.e., $a_{2k+1}(q) = 0$ and $a_2(q) = 1/2$; then the Laplace argument yields for all $T \geqslant T_c$ that the distribution $\phi(t, \beta, q, 0)$ exists and is not trivial if and only if $\delta = 0$; in that case, it is given by

$$\phi(t, \beta, q, 0) = \exp\left\{-a_2(q)\frac{t^2}{2}\right\} \tag{9}$$

we have proved the following result.

Theorem 2.1. If $q \neq 0$, i.e., $a_1(q) = 0$ and $a_2(q) = 1/2$, then the probability distribution of the fluctuation $F^q_0(\sigma)$ is normal ($\delta = 0$) and of Gaussian type [formula (9)] for all values of $T \geqslant T_c$.

This theorem generalizes the particular case treated in ref. 4, which can be understood in the above setting by choosing q (e.g., $q=\pi$) in the interval $x \in \Lambda = [1, 2N]$ such that for an equal number of lattice points $\cos qx = 1$ and $\cos qx = -1$, i.e., $a_{1,2N}(q) = 0$ for all N. For this choice of q one gets

$$F^{\pi}_{\delta,2N}(\sigma) = \frac{1}{2^{1/2+\delta}} [F^{0}_{\delta,N}(\sigma) - \tilde{F}^{0}_{\delta,N}(\sigma)]$$

Therefore, Theorem 2.1 yields that the subsystems fluctuate coherently; $F^{\pi}_{0}(\sigma)$ is Gaussian in spite of the fact that $F^{0}_{0,N}(\sigma)$ and $\tilde{F}^{0}_{0,N}(\sigma)$ blow up as $N \to \infty$ (see Theorem 2.2).

In the language of physics, it should be remarked that the variance of the above fluctuation is the Fourier transform of the susceptibility at the point q. Heuristic physical arguments lead to the property that this susceptibility is always finite for $q \neq 0$, even at the critical point. Theorem 2.1 is a first step toward a general rigorous proof of this property, because $a_1(q) = 0$, if $q \neq 0$. It shows self-canceling of coherent non-Gaussian fluctuations corresponding to different parts of the system.

Let us now compute the characteristic function (8) in the case $a_1(q) \neq 0$, i.e., $q = 0$. Computing again formula (8), one gets for $T > T_c$ a nontrivial result if and only if $\delta = 0$,

$$\phi(t, \beta, q, 0) = \exp\left\{\frac{-t^2}{2}\left[\frac{\beta J a_1^2(q)}{1-\beta J} + a_2(q)\right]\right\} \tag{10}$$

On the other hand, if $T = T_c$, then the limit (8) exists and is nontrivial if and only if $\delta = 1/4$. It is given by

$$\phi(t, \beta, q, 0) = \frac{\int du \exp[-(u^4/12) + itua_1(q)]}{\int du \exp(-(u^4/12))} \tag{11}$$

We have obtained the following results:

Theorem 2.2. If $q = 0$, i.e., $a_1(q) = a_2(q) = 1$, then the probability distribution of the random variable for the limit variable of

$$F^{q}_{\delta,\Lambda}(\sigma) = \frac{1}{N^{1/2+\delta}} \sum_{x \in \Lambda} \sigma_x \cos qx$$

is normal ($\delta = 0$) and Gaussian (10) if $T > T_c$; it is abnormal ($\delta = 1/4$) and non-Gaussian (11) if $T = T_c$. ■

The content of this theorem is in fact the case discussed in refs. 1 and 2. We discussed only cases where q is fixed. However, the considerations show that also the case $q_N \to 0$ can be considered. In that case there might be an interference between the quantities $a_{1,N}(q_N)$ and $a_{2,N}(q_N)$. This is even more true if we take into account a fading out external field, $h_x(N) \to 0$.

In the next section we discuss the q-mode fluctuation of the magnetization depending on the external parameters h_x, $x \in \Lambda$.

3. THE MAGNETIC FIELD ($h \to 0$)

In this case we put

$$h_x = \frac{\hat{h}}{N^\alpha}, \qquad \alpha > 0 \quad \text{for all} \quad x \in \Lambda \tag{12}$$

i.e., we let the external magnetic field tend to zero following a power law in the volume, keeping the value of $\hat{h}$ constant. We will distinguish here different cases depending on the value of the parameter α. One can imagine that if α is large enough, the effect of the external field is vanishing, but for α small enough, there might be an influence on the critical index δ as well as on the limit probability distribution. In fact that is what we are going to demonstrate.

After substitution of (12) in the partition function (1) one gets

$$Z_\Lambda(\beta, \hat{h}N^{-\alpha}) = \left(\frac{N}{2\pi}\right)^{1/2} 2^N \int dv \exp\{-Nf(v, \beta, N^\alpha)\} \tag{13}$$

where

$$f(v, \beta, \hat{h}N^{-\alpha}) = \frac{v^2}{2} - \ln ch \left[v(\beta J)^{1/2} + \frac{\beta \hat{h}}{N^\alpha} \right] \tag{14}$$

For fixed N, consider the variational problem, related to the Laplace arguments for (13):

$$\inf_v f(\beta, v, \hat{h}N^{-\alpha}) = f(\beta, \bar{v}, \hat{h}N^{-\alpha}) \tag{15}$$

where $\bar{v} = \bar{v}(\beta, \hat{h}N^{-\alpha})$ is its solution. The value $\bar{v}$ does depend on the volume parameter N^α, while

$$\lim_N \bar{v}(\beta \leqslant \beta_c, \hat{h}N^{-\alpha}) = \lim_N (\beta J)^{1/2} m_\Lambda = 0$$

For the random variable (2) we compute again the characteristic function (3):

$$\phi(t, \beta, q, \hat{h}) \equiv \lim_{N\to\infty} Z_\Lambda^{-1}(\beta, \hat{h}) \left(\frac{N}{2\pi}\right)^{1/2} \int dv \exp\left\{-\frac{Nv^2}{2} + \sum_{x\in\Lambda} \ln 2\,\mathrm{ch}\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha} + \frac{it}{N^{1/2+\delta}}\cos qx\right] - \frac{it}{N^{-1/2+\delta}} m_\Lambda a_{1,N}(q)\right\} \tag{16}$$

where $Z_\Lambda(\beta, \hat{h})$ is given by formula (13).

As in Section 2, we can distinguish the two cases $q \neq 0$ and $q = 0$.

Again we start with the case of the wave vector $q \neq 0$, i.e., $a_1(q) = 0$ and $a_2(q) > 0$. We expand the ln $ch[\cdots]$ in (16) around the value $v(\beta J)^{1/2}$ and get

$$\phi(t, \beta, q, \hat{h}) = \lim_N \left(\int dv \exp\left\{-N\left[\frac{v^2}{2} - \ln ch\, v(\beta J)^{1/2}\right] + [th\, v(\beta J)]\, \hat{h}\beta N^{1-\alpha} + \frac{1}{2}[ch^{-2}\, v(\beta J)^{1/2}](\beta\hat{h})^2 N^{1-2\alpha} + \cdots\right\}\right)^{-1}$$
$$\times \int dv \exp\left\{-N\left[\frac{v^2}{2} - \ln ch\, v(\beta J)^{1/2}\right] + [th\, v(\beta J)^{1/2}]\, \beta\hat{h}\, N^{1-\alpha} + \frac{1}{2}[ch^{-2}\, v(\beta J)^{1/2}] \times [(\beta\hat{h})^2 N^{1-2\alpha} + (it)^2 N^{-2\delta} a_{2,N}(q)] + \cdots\right\}$$

If $T > T_c$, using (4) and the transformation $v\sqrt{N} = u$, one checks that the limit (16) exists if and only if $\delta = 0$ for all values of $\alpha > 0$ and that it is given by

$$\phi(t, \beta, q, \hat{h}) = \exp\left[-\frac{t^2}{2} a_2(q)\right] \tag{17}$$

If $T = T_c$, using (5) and the transformation $vN^{1/4} = u$, one checks the existence of the limit (16) if and only if $\delta = 0$, again for all values of $\alpha > 0$, and that it is given again by the expression (17). we have proved the following result (cf. Theorem 2.1):

Theorem 3.1. If $q \neq 0$, i.e., $a_1(q) = 0$ and $a_2(q) = 1/2$, then the limit probability distribution of the magnetization in an external field $h_x = \hat{h}/N^\alpha$,

$\alpha > 0$, is normal ($\delta = 0$) and Gaussian (17) for all values of $\alpha > 0$ and all temperatures $T \geqslant T_c$. ▮

Next we analyze the case $q = 0$, i.e., $a_1(q) \neq 0$. Here we distinguish the cases $T > T_c$ and $T = T_c$. First we treat the case $T > T_c$.

It is immediately checked that the solution $\bar{v}$ of the variational problem (15) yields

$$\bar{v} = (\beta J)^{1/2}\, th\left[\bar{v}(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right] \tag{18}$$

Now, as $T > T_c$ or $\beta J < 1$, this gives the following expansion of $\bar{v}$ for large N:

$$\bar{v}(1 - \beta J) = \frac{\beta\hat{h}}{N^\alpha}(\beta J)^{1/2} + \cdots \tag{19}$$

and $m_\Lambda(1 - \beta J) = \beta(\hat{h}/N^\alpha) + \cdots$. Therefore, one gets

$$th\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right] - th\left[\bar{v}(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^a}\right] = (\beta J)^{1/2}(v - \bar{v}) + \cdots$$

The numerator of (16) has then the following expansion:

$$\begin{aligned} \int dv \exp\Bigg(&-N\left\{\frac{v^2}{2} - \ln ch\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^a}\right]\right\} \\ &+ \left\{th\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right]\right\} itN^{1/2-\delta}a_{1,N}(q) \\ &- \frac{t^2}{2}\left\{ch^{-2}\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right]\right\} N^{-2\delta}a_{2,N}(q) + \cdots \\ &- itm_\Lambda N^{1/2-\delta}a_{1,N}(q)\Bigg) \end{aligned} \tag{20}$$

Using also the expansion

$$\begin{aligned} &\frac{v^2}{2} - \ln ch\left[v(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right] - \frac{\bar{v}^2}{2} + \ln ch\left[m_\Lambda(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right] \\ &\quad = \frac{(v - \bar{v})^2}{2}\left\{1 - \beta J\, ch^{-2}\left[\bar{v}(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right]\right\} + \cdots \end{aligned}$$

and the rescaling $\sqrt{N}\,u = v - \bar{v}$, we find that Eq. (16) becomes equal to, for $\varepsilon > 0$,

$$\phi(t, \beta, q, \hat{h}) = \lim_N \left(\int_{-(\varepsilon+\bar{v})\sqrt{N}}^{(\varepsilon-\bar{v})\sqrt{N}} du \right.$$

$$\left. \times \exp\left\{ -\frac{u^2}{2}\left[1 - \beta J\, ch^{-2}\left(\frac{\beta\hat{h}}{N^\alpha(1-\beta J)}\right)\right]\right\}\right)^{-1}$$

$$\times \int_{-(\varepsilon+\bar{v})\sqrt{N}}^{(\varepsilon-\bar{v})\sqrt{N}} du\, exp\left\{ -\frac{u^2}{2}\left[1 - \beta J\, ch^{-2}\left(\frac{\beta\hat{h}}{N^\alpha(1-\beta J)}\right)\right]\right.$$

$$+ itN^{-\delta}(\beta J)^{1/2}\, a_{1,N}(q)\, u - \frac{t^2}{2}\frac{a_{2,N}(q)}{N^{2\delta}}$$

$$\left. \times\, ch^{-2}\left[\left(\frac{u}{\sqrt{N}} + \bar{v}\right)(\beta J)^{1/2} + \frac{\beta\hat{h}}{N^\alpha}\right] + \cdots \right\}$$

Using (19) and again the Laplace argument, one gets that this limit exists for all $\alpha > 0$, and is not trivial if and only if $\delta = 0$. It is given by the normal Gaussian characteristic function

$$\phi(t, \beta, q, \hat{h}) = \exp\left\{ -\frac{t^2}{2}\left[\frac{\beta J a_1^2(q)}{1-\beta J} + a_2(q)\right]\right\} \tag{21}$$

i.e., we have proved the following expected result (cf. Theorem 2.2):

Theorem 3.2. If q is such that $a_1(q) \neq 0$, then the limit probability distribution of the fluctuation $F_\delta^q(\sigma)$ in the scaled external field $h_x = \hat{h}/N^\alpha$, $\alpha > 0$, is normal ($\delta = 0$) and Gaussian (21) for all values of $\alpha > 0$ and for all temperatures $T > T_c$. ▮

We are left with the case $T = T_c$ or $\beta_c J = 1$. With respect to the previous case there are some essential changes. We start with the expansion of the free energy functional (14):

$$f(v, \beta_c, \hat{h}N^{-\alpha}) = f(\bar{v}, \beta_c, \hat{h}N^{-\alpha}) + \frac{1}{2}(v-\bar{v})^2 D_N^{(2)}$$

$$+ \frac{1}{3!}(v-\bar{v})^3 D_N^{(3)} + \frac{1}{4!}(v-\bar{v})^4 D_N^{(4)} + \cdots \tag{22}$$

where

$$D_N^{(2)}(\alpha) = 1 - ch^{-2}\left(\bar{v} + \frac{\beta_c\hat{h}}{N^\alpha}\right) \tag{23}$$

$$D_N^{(3)}(\alpha) = -2ch^{-2}\left(\bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right) th\left(\bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right) \tag{24}$$

$$D_N^{(4)}(\alpha) = 2\, ch^{-4}\left(\bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right) - 4\, ch^{-2}\left(\bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right) \text{th}^2\left(\bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right) \tag{25}$$

...

As now $1 = \beta_c J$, the first-order expansion of Eq. (18) is no longer valid. Instead of Eq. (19) one gets now

$$\bar{v} = \left(\frac{3\beta_c \hat{h}}{N^\alpha}\right)^{1/3} + \cdots \tag{26}$$

Using this equation for $\bar{v}$ as a function of N, one can derive the behavior of the coefficient $D_N^{(2)}(\alpha)$. One finds

$$D_N^{(2)}(\alpha) = \left(\frac{3\beta_c \hat{h}}{N^\alpha}\right)^{2/3} + \cdots \tag{27}$$

for large N. In view of the applicability of the Laplace argument, one has to look at the behavior $\lim_{N \to \infty} ND_N^{(2)}(\alpha)$. The question is whether the quadratic term in (22) is dominant or not. If $ND_N^{(2)}(\alpha) \to \infty$ for $N \to \infty$, then the quadratic term is dominant; it is clear that this fact depends on the value of α. If $\alpha < 3/2$, then $ND_N^{(2)}(\alpha) \to \infty$. Let us first treat this case.

Using the expansions (20) and (22), and the fact that the quadratic term is dominant, together with the scaling $(ND_N^{(2)})^{1/2}\,(v - \bar{v}) = u$, we find that the limit (16) becomes for any $\varepsilon > 0$

$$\begin{aligned}
&\phi(t, \beta_c, q, \hat{h}) \\
&\quad = \lim_N \left[\int_{-(ND_N^{(2)})^{1/2}(\varepsilon + \bar{v})}^{(ND_N^{(2)})^{1/2}(\varepsilon - \bar{v})} du\, e^{-u^2/2}\right]^{-1} \int_{-(ND_N^{(2)})^{1/2}(\varepsilon + \bar{v})}^{(ND_N^{(2)})^{1/2}(\varepsilon - \bar{v})} du \\
&\quad \times \exp\left\{-\frac{u^2}{2} + \frac{itua_1(q)}{N^\delta (D_N^{(2)})^{1/2}} - \frac{t^2}{2}\right. \\
&\quad \left.\times \left[ch^{-2}\left(\frac{u}{(ND_N^{(2)})^{1/2}} + \bar{v} + \frac{\beta_c \hat{h}}{N^\alpha}\right)\right] \frac{a_2(q)}{N^{2\delta}} + \cdots\right\}
\end{aligned}$$

By (27) one gets a nontrivial limit distribution if the index δ is such that $N^\delta (D_N^{(2)})^{1/2} \simeq N^{\delta - \alpha/3}$ tends to a nonzero constant, i.e., if and only if $\delta = \alpha/3$. The limit is given by

$$\phi(t, \beta_c, q, \hat{h}) = \exp\left\{-\frac{t^2}{2} \frac{a_1^2(q)}{(3\beta_c \hat{h})^{2/3}}\right\} \tag{28}$$

Therefore, we have proved the following result:

Theorem 3.3. If $q=0$, i.e., $a_1(q)=a_2(q)=1$, then the limit distribution of the q-mode magnetization fluctuation in the scaled external field $h_x=\hat{h}/N^\alpha$, $\alpha>0$, at $T=T_c$ is abnormal with $\delta=\alpha/3$ for all $\alpha<3/2$, but Gaussian (28). ▌

Remark that the critical index δ takes all values between zero and one-half, depending linearly on the parameter α. This means that the degree δ of criticality is influenced by the scaling of the external field. Remark also that the limit probability distribution is Gaussian for $\alpha<3/2$. The presence of the magnetic field h_x generates the Gaussian character, if it does not fall off too fast; the parameter α seems to have a critical value $\alpha_c=3/2$ for *Gaussianity*. This will become clear from what follows.

Now we deal with the case that $\alpha \geqslant 3/2$. Using again the expansions (20) and (22), performing the rescaling $N(v-\bar{v})^4=u^4$, and using the fact that $\lim_N D_N^{(4)}=2$, we find that the limit (16) can be written as

$$\phi(t,\beta_c,q,\hat{h}) = \lim_N \frac{\left(\begin{array}{c}\int du \exp\{-\frac{1}{2}u^2\sqrt{N}D_N^{(2)}-(u^3/3!)N^{1/4}D_N^{(3)}-\frac{1}{12}u^4 \\ +\cdots+ itua_1(q)N^{1/4-\delta}\}\end{array}\right)}{\int du \exp\{-\frac{1}{2}u^2\sqrt{N}D_N^{(2)}-(u^3/3!)N^{1/4}D_N^{(3)}-\frac{1}{12}u^4+\cdots\}} \tag{29}$$

Note that if $\alpha>3/2$, then

$$\lim_N \sqrt{N}D_N^{(2)}=\lim_N N^{1/2-2\alpha/3}=0$$

$$\lim_N N^{1/4}D_N^{(3)}=\lim_N N^{1/4-\alpha/3}=0$$

and the coefficients of the terms of order higher than four also vanish in (29). Therefore, the limit (29) exists and yields a nontrivial probability distribution if and only if $\delta=1/4$. The limit is given by [cf. (11)]:

$$\phi(t,\beta_c,q,\hat{h})=\frac{\int du \exp\{-(u^4/12)+itua_1(q)\}}{\int du \exp\{-(u^4/12)\}} \tag{30}$$

We have proved the following result:

Theorem 3.4. For $q=0$, i.e., $a_1(q)=1$, the limit probability distribution of the q-mode magnetization fluctuation in the scaled external field $h_x=\hat{h}/N^\alpha$ with $\alpha>3/2$ at $T=T_c$ is abnormal with $\delta=1/4$ and non-Gaussian (30). ▌

This theorem yields the same result as in the case $h=0$ (see refs. 1 and 2 and Theorem 2.2) and this for all values of $\alpha>3/2$. If the external field drops off with the volume fast enough, the influence of the external field is nonexistent. Remark that the index $\delta = 1/4$, indicating the degree of criticality, is here independent of α as long as $\alpha > 3/2$.

Finally it remains to look at the marginal or critical value $\alpha_c = 3/2$. In this case one has

$$\lim_N \sqrt{N}\, D_N^{(2)} = A_2 = (3\beta_c \hat{h})^{2/3}$$

$$\lim_N N^{1/4} D_N^{(3)} = A_3 = (3\beta_c \hat{h})^{1/3}$$

and the coefficient of the terms of order greater than four vanish in (29) in the limit $N \to \infty$. Again the limit (29) exists and is nontrivial if and only if $\delta = 1/4$; it is now given by

$$\phi(t, \beta_c, q, \hat{h}) = \frac{\int du \exp\{-(u^2/2) A_2 + (u^3/3!) A_3 - u^4/12 + itua_1(q)\}}{\int du \exp\{-(u^2/2) A_2 + (u^3/3!) A_3 - u^4/12\}} \tag{31}$$

We have proved the following result:

Theorem 3.5. If $q = 0$, such that $a_1(q) = 1$, the limit probability distribution of the q-mode magnetization fluctuation in the scaled external field $h_x = \hat{h}/N^{3/2}$ at $T = T_c$ is abnormal with $\delta = 1/4$ and non-Gaussian, defined by the expression (31).

4. THE LONG-WAVELENGTH LIMIT ($q \to 0$)

The long-wavelength limit is the natural physical language to describe the critical susceptibility $\lim_{q \to 0} \chi(T_c, q) = \infty$.

As above, suppose that this limit is taken by the rate exponent $\gamma \geqslant 0$ and the amplitude $\hat{q}$, i.e., take $q_N \equiv \hat{q} N^{-\gamma}$. Then one gets that for $N \to \infty$

$$a_{1,N}(q_N) = \begin{cases} 1 + c_1 N^{-\Gamma_1(\gamma)} + \cdots; & \gamma > 1 \quad [\Gamma_1(\gamma > 1) > 0] \\ (\sin \hat{q})/\hat{q} + c_2 N^{-\Gamma_2(\gamma)} + \cdots; & \gamma = 1 \quad [\Gamma_2(\gamma = 1) > 0] \\ N^{-(1-\gamma)} \sin(\hat{q} N^{1-\gamma}) + \cdots; & \gamma < 1 \end{cases} \tag{32}$$

One easily gets the asymptotics for $a_{2,N}(q_N)$, $a_{3,N}(q_N)$, etc., from the trigonometric formulas. Let $\lim_N a_{1,N}(q_N) \equiv \hat{a}_1(\hat{q})$. Then we get [cf. (6)]

$$\begin{aligned} \hat{a}_2(\hat{q}) &\equiv \lim_N a_{2,N}(q_N) = \tfrac{1}{2}[1 + \hat{a}_1(2\hat{q})] > 0 \\ \hat{a}_3(\hat{q}) &\equiv \lim_N a_{3,N}(q_N) = \tfrac{1}{4}[3\hat{a}_1(\hat{q}) + \hat{a}_1(3\hat{q})], \quad \text{etc.} \end{aligned} \tag{33}$$

Now we can use to advantage our notations in Section 2 to calculate the limiting characteristic function

$$\hat{\phi}(t, \beta, \hat{q}, 0) \equiv \lim_N \mu^{\Lambda}_{\beta,0}(\exp[itF^{q_N}_{\delta,\Lambda}(\sigma)])$$

for $T \geqslant T_c$. Using the asymptotics (32) and (33) in the formula (8), we get for $T > T_c$ [cf. (10)]

$$\hat{\phi}(t, \beta, \hat{q}, 0) = \exp\left\{\frac{-t^2}{2}\left[\frac{\beta J \hat{a}_1^2(\hat{q})}{1-\beta J} + \hat{a}_2(\hat{q})\right]\right\} \tag{34}$$

while for $T = T_c$ we have to distinguish three cases, yielding

$$\hat{\phi}(t, \beta_c, \hat{q}, 0) = \begin{cases} \exp\left\{-\dfrac{t^2}{2}\hat{a}_2(\hat{q})\right\}; \\ \qquad \gamma < \dfrac{3}{4} \quad (\delta = 0) \\ \text{“}\lim\limits_N \dfrac{\int du \exp\{-(u^4/12) + itu \sin(\hat{q}N^{1-\gamma})\}}{\int du \exp\{-u^4/12\}}\text{”}; \\ \qquad \dfrac{3}{4} \leqslant \gamma < 1 \quad \left[\delta = \dfrac{1}{4} - (1-\gamma)\right] \\ \dfrac{\int du \exp\{-(u^4/12) + itu\hat{a}_1(\hat{q})\}}{\int du\{-u^4/12\}}; \\ \qquad \gamma \geqslant 1, \quad \hat{q} \neq n\pi, \quad n \in \mathbb{Z}^1 \\ \qquad \left(\delta = \dfrac{1}{4}\right) \\ \exp\left\{-\dfrac{t^2}{2}\hat{a}_2(\hat{q})\right\}; \\ \qquad \gamma = 1, \quad \hat{q} = n\pi, \quad n \in \mathbb{Z}^1\backslash\{0\} \\ \qquad (\delta = 0) \end{cases} \tag{35}$$

It is clear that (34), (35), with (32), (33) for $\hat{q} \in \mathbb{R}$ interpolate the cases (9) and (11) for $\gamma < 3/4$ and $\gamma \geqslant 1$. Therefore, the statements of Theorems 2.1 and 2.2 are valid in a more general situation $q_N \to 0$, but now for $\hat{\phi}(t, \beta, \hat{q}, 0)$. For $\gamma = 1$ and $\hat{q} = n\pi (n \in \mathbb{Z}^1\backslash\{0\})$ we get again a "Papangelou case,"[4] the fluctuations are normal ($\delta = 0$). Again they are the difference of two coherent and compensating [sin $\hat{q} = 0$, (32)] abnormal fluctuations of two subsystems; see also Theorem 2.1 and the remarks following it. For $3/4 \leqslant \gamma < 1$ we have *no limit in* (35), *except for subsequences.* This is an indication of an "essential singularity" for the long-wavelength magnetic fluctuations at $T = T_c$. The situation gets even more complicated for the fading out external field.

As long as $\gamma \geqslant 1$, we can easily generalize the statement of Theorem 3.2 [cf. (21)]:

$$\hat{\phi}(t, \beta, \hat{q}, \hat{h}) = \exp\left\{-\frac{t^2}{2}\left[\frac{\beta J \hat{a}_1^2(\hat{q})}{1-\beta J} + \hat{a}_2(\hat{q})\right]\right\} \tag{36}$$

The same formula covers the case $\gamma < 1$ for $T > T_c$ [cf. (17)], i.e., part of Theorem 3.1.

Using the arguments of Theorem 3.3, one can check this statement for $\gamma \geqslant 1$ at $T = T_c$:

$$\hat{\phi}(t, \beta_c, \hat{q}, \hat{h}) = \exp\left\{-\frac{t^2}{2}\frac{\hat{a}_1^2(\hat{q})}{(3\beta_c \hat{h})^{2/3}}\right\} \tag{37}$$

In this case $\delta = \alpha/3$ $(\alpha < 3/2)$ [cf. (28)].

But for $\gamma < 1$ the situation changes drastically:

(a) Let $\alpha < 3/2$ and $\alpha/3 + \gamma - 1 > 0$. Then one gets for $\delta = \alpha/3 + \gamma - 1$ [cf. (28) and (32)] that

$$\hat{\phi}(t, \beta_c, \hat{q}, \hat{h}) = \text{“}\lim_N \exp\left\{-\frac{t^2}{2}\frac{\sin^2(\hat{q}N^{1-\gamma})}{(3\beta_c \hat{h})^{2/3}}\right\}\text{”} \tag{38}$$

This means that we have *no limit* for the characteristic function of the magnetic fluctuations *but different limits* for subsequences $\{N_j \to \infty\}$.

For $\alpha < 3/2$ and $\alpha/3 + \gamma - 1 < 0$, one gets [cf. (17)]

$$\hat{\phi}(t, \beta_c, \hat{q}, \hat{h}) = \exp\left\{-\frac{t^2}{2}\hat{a}_2(\hat{q})\right\} \tag{39}$$

with $\hat{a}_2(\hat{q}) = 1/2$.

(b) Let $\alpha > 3/2$. Then using (29), we get the first two cases described by (35).

(c) Let $\alpha = 3/2$. Then by formula (31) and the arguments used in (35) we again get the two first cases indicated there, $\gamma < 3/4$ and $3/4 \leqslant \gamma < 1$, but for the distribution function defined by A_2 and A_3 [cf. (31) and (35)].

5. CONCLUDING REMARKS

Theorems 3.3–3.5 indicate very explicitly the importance of the scaling parameter α of the external field, which plays the role of a boundary condition. For $\alpha > 3/2$ the influence of the field is nonexistent; for $\alpha < 3/2$ the critical index δ characterizing the abnormality of the fluctuations depends on α. Such a phenomenon was also already observed in the study of quantum systems.[8] Finally $\alpha_c = 3/2$ is a critical value.

As is also clear from the above, different mean-field-type models can be characterized by different rate functions. On the other hand, for mean fields there is a generally accepted form of rate function, usually called the Ginzburg–Landau rate function. In spite of the fact that the latter yields the same criticality (e.g., also $\alpha_c = 3/2$), the distributions of the random variables, like fluctuations, are different. In particular, the rigorous Curie–Weiss model computations obtained above yield a different characteristic function than the Ginzburg–Landau one. Typical for the latter is that it is an even rate function defining the criticality, whereas formula (31) does not yield an even one.

Finally, we did not discuss the region $T < T_c$. In this case one can use the above techniques, but now with $h_x = \pm(\hat{h}/N^{\alpha})$, in order to recover the two extremal measures (see, e.g., ref. 9). One gets straightforwardly that the q-mode magnetization fluctuations are always normal and Gaussian.

The long-wavelength limit is now the next interesting question (i.e., $q \to 0$). There are many ways to take this limit. In Section 4 we gave a first discussion of it, by taking $q_N = \hat{q}N^{-\gamma}$, $N \to \infty$ and $\hat{q}$ fixed. Already we find there that for γ satisfying $3/4 \leqslant \gamma < 1$ there is no limit distribution except for subsequences. The combination of the long-wavelength limit ($\hat{q}N^{-\gamma}$) with the vanishing boundary condition ($\hat{h}N^{-\alpha}$) at $T = T_c$ indicates already the richness of the structure. Here also the situation is not completely cleared up and calls for elaboration. It remains an interesting point.

REFERENCES

1. R. S. Ellis and C. M. Newman, *Z. Wahrsch. Verw. Geb.* **44**:117–139 (1978).
2. R. S. Ellis and C. M. Newman, *J. Stat. Phys.* **19**:149–161 (1978).
3. M. Fannes, A. Kossakowski, and A. Verbeure, *J. Stat. Phys.* **65**:801–811 (1991).
4. F. Papangelou, *Prob. Theory Related Fields* **83**:265–278 (1989).
5. A. Verbeure and V. A. Zagrebnov, Critical quantum fluctuations for a displacement phase transition, in preparation.
6. M. Cassandro and G. Jona-Lasinio, *Adv. Phys.* **27**:913–941 (1978).
7. C. M. Newman and D. L. Stein, *Phys. Rev. B* **46**:973–982 (1992).
8. A. Verbeure and V. A. Zagrebnov, *J. Stat. Phys.* **69**:329–359 (1992).
9. J. G. Brankov, N. S. Tonchev, and V. A. Zagrebnov, *Theor. Math. Phys.* **66**:72–80 (1986).